

Lagrange Interpolation Based at the Zeros of Orthonormal Polynomials with Freud Weights

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continuous function f vanishing outside a finite interval

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} [|f(x) - L_n(f; x)|^p W(x)] dx = 0$$

holds, then we have

$$\int_{-\infty}^{\infty} [W_Q^{-1}(x)/(1 + |x|)]^p W(x) dx < \infty.$$

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1. INTRODUCTION

Let Q be an even, continuous, and real-valued function defined on the real line $\mathbb{R} = (-\infty, \infty)$, and let $Q' \in C(\mathbb{R})$, $Q'(x) > 0$ on $(0, \infty)$, and Q'' be continuous on $(0, \infty)$. Furthermore, we assume that for certain constants $1 < A \leq B$,

$$A \leq \{(d/dx)(xQ'(x))\}/Q'(x) \leq B, \quad x \in (0, \infty).$$

We call the function $Q(x)$ a Freud exponent, and then we consider what is called a Freud weight

$$W_Q^2(x) = \exp\{-Q(x)\}. \quad (1.1)$$

We note that if $\alpha > 1$, then $W_\alpha^2(x) = \exp(-|x|^\alpha)$ is a Freud weight. The Mhaskar–Rahmanov–Saff number a_u is defined as the positive root of the equation

$$u = (2/\pi) \int_0^1 a_u t Q'(a_u t) (1 - t^2)^{-(1/2)} dt, \quad u > 0.$$

The number a_u plays an important role in the study of the approximation theory. Let Π_n denote the class of real polynomials of degree at most n , and let $\{p_n(x)\} = \{p_n(W_Q^2; x)\}$, $p_n \in \Pi_n$ be the sequence of orthonormal polynomials with respect to W_Q^2 , that is,

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) W_Q^2(x) dx = \delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

We denote the zeros of $P_n(x)$ by x_{kn} , $k = 1, 2, \dots, n$, where

$$x_{nn} < x_{n-1, n} < \dots < x_{1n}.$$

Then, for a given $f \in C(\mathbb{R})$ the Lagrange interpolation polynomial $L_n(f)$ based at the zeros $\{x_{kn}\}$ of $P_n(x)$ is defined to be a unique polynomial in Π_{n-1} such that

$$L_n(f; x_{kn}) = f(x_{kn}), \quad k = 1, 2, \dots, n.$$

Nevai obtained the following.

NEVAI'S THEOREM ([12]). *Let $L_n(f; x)$ denote the Lagrange interpolation polynomial at the zeros of $P_n(W_2^2; x)$ for the weight $W_2^2(x) = \exp(-x^2)$. Let $W(\geq 0) \in L_1(\mathbb{R})$ and $0 < p < \infty$ be given. Suppose that for every continuous function f vanishing outside a finite interval,*

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} |f(x) - L_n(f; x)|^p W(x) dx = 0.$$

Then

$$\int_{-\infty}^{\infty} [\exp(x^2/2)/(1 + |x|)]^p W(x) dx < \infty.$$

In this paper we extend Nevai's Theorem for the Freud weight (1.1).

THEOREM. *Let $W(\geq 0) \in L_1(\mathbb{R})$ and $0 < p < \infty$ be given. If for every continuous function f vanishing outside a finite interval*

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} [|f(x) - L_n(f; x)|^p W(x)] dx = 0 \quad (1.2)$$

holds, then we have

$$\int_{-\infty}^{\infty} [W_Q^{-1}(x)/(1 + |x|)]^p W(x) dx < \infty. \quad (1.3)$$

If in the theorem we consider especially the case of $W(x) = W_Q^p(x)(1 + |x|)^{-\Delta p}$, $\Delta > 1/p - 1$, then obviously we have (1.3). In this case, that is, $W(x) = W_Q^p(x)(1 + |x|)^{-\Delta p}$, Lubinsky and Matijla have obtained a complete solution as follows.

THEOREM OF LUBINSKY-MATIJLA ([9]). *Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$, and $\hat{\alpha} = \min(1, \alpha)$. Then for*

$$\lim_{n \rightarrow \infty} \|\{f(x) - L_n(f; x)\} W_Q(x)(1 + |x|)^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous function $f \in C(\mathbb{R})$ satisfying

$$\lim_{n \rightarrow \infty} |f(x)| W_Q(x)(1 + |x|)^{\alpha} = 0,$$

if $p \leq 4$, it is necessary and sufficient that

$$\Delta > -\hat{\alpha} + 1/p;$$

and if $p > 4$ and $\alpha \neq 1$, it is necessary and sufficient that

$$a_n^{1/p - (\hat{\alpha} + \Delta)} n^{(1/6)(1 - 4/p)} = O(1), \quad n \rightarrow \infty;$$

and if $p > 4$ and $\alpha = 1$, it is necessary and sufficient that

$$a_n^{1/p - (\hat{\alpha} + \Delta)} n^{(1/6)(1 - 4/p)} = O(1/\log n), \quad n \rightarrow \infty.$$

Our theorem asserts that if for a certain $W(\geq 0) \in L_1(\mathbb{R})$,

$$\int_{-\infty}^{\infty} [W_Q^{-1}(x)/(1 + |x|)]^p W(x) dx = \infty$$

holds, then for a continuous function f we see that $L_n(f)$ does not converge to f :

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} [|f(x) - L_n(f; x)|^p W(x)] dx \neq 0.$$

2. FUNDAMENTAL LEMMAS

Throughout this paper c will denote a positive constant independent of n and x , and the letter c will denote a constant which may differ at each different occurrence, even in the chain of inequalities. Let $c(a, b, \dots)$ mean a constant depending on a, b, \dots . By $f(x) \sim g(x)$ we denote $c_1 \leq f(x)/g(x) \leq c_2$ for certain positive constants c_1, c_2 , and for all relevant x .

LEMMA 2.1. *Let $|x| \leq ca_n$ for certain constants $c > 0$. Then we have*

$$W_Q(x) \sim W_Q(x') \quad \text{for } |x - x'| \leq \kappa a_n/n. \quad (2.1)$$

Proof. For $|x - x'| \leq \kappa a_n/n$ ($|x|, |x'| \leq ca_n$) we see

$$\begin{aligned} |Q(x) - Q(x')| &\leq |Q'(\xi)| \kappa a_n/n \quad (x < \xi < x') \\ &\leq |Q'(ca_n)| \kappa a_n/n \\ &\leq c(\kappa) \end{aligned}$$

by [6, Lemma 5.1(c)]. Consequently, we see

$$\begin{aligned} W_Q(x)/W_Q(x'), W_Q(x')/W_Q(x) &\leq \exp \{|Q(x) - Q(x')|\} \\ &\leq \exp \{c(\kappa)\}, \end{aligned}$$

that is, (2.1) follows. ■

Let $\{p_n(x)\} = \{p_n(W_Q^2; x)\}$ be the orthonormal polynomials with respect to W_Q^2 , and let $r_n = \gamma_{n-1}/\gamma_n$, where γ_n is the leading coefficient of $p_n(x)$, that is, $p_n(x) = \gamma_n x^n + \dots$. The following lemma is useful for an estimate of values of $p'_n(x)$.

LEMMA 2.2 (cf. [2, Theorem 5], [10, Theorem 3.2]). *We have an equation*

$$p'_n(x) = A_n(x) p_{n-1}(x) - B_n(x) p_n(x),$$

where

$$A_n(x) = r_n \int_{-\infty}^{\infty} p_n^2(t) \bar{Q}(x, t) W_Q^2(t) dt,$$

$$B_n(x) = r_n \int_{-\infty}^{\infty} p_n(t) p_{n-1}(t) \bar{Q}(x, t) W_Q^2(t) dt,$$

and

$$\bar{Q}(x, t) = \{Q'(x) - Q'(t)\}/(x - t).$$

Proof. We can write $p'_n(x)$ in the Fourier expansion in terms of the reproducing kernel $K_n(x, t)$ as

$$p'_n(x) = \int_{-\infty}^{\infty} p'_n(t) K_n(x, t) W_Q^2(t) dt, \quad (2.2)$$

where

$$\begin{aligned} K_n(x, t) &= \sum_{k=0}^{n-1} p_k(x) p_k(t) \\ &= r_n \{p_n(x) p_{n-1}(t) - p_n(t) p_{n-1}(x)\} / (x - t). \end{aligned} \quad (2.3)$$

Since $\int_{-\infty}^{\infty} p_n(t) \{(d/dt) K_n(x, t)\} W_Q^2(t) dt = 0$, by (2.2) and (2.3)

$$\begin{aligned} p'_n(x) &= - \int_{-\infty}^{\infty} p_n(t) K_n(x, t) \{(d/dt) W_Q^2(t)\} dt \\ &= \int_{-\infty}^{\infty} p_n(t) K_n(x, t) Q'(t) W_Q^2(t) dt \\ &= -r_n \int_{-\infty}^{\infty} p_n(t) \{p_n(x) p_{n-1}(t) - p_n(t) p_{n-1}(x)\} \\ &\quad \times \bar{Q}(x, t) W_Q^2(t) dt \\ &= \left\{ r_n \int_{-\infty}^{\infty} p_n^2(t) \bar{Q}(x, t) W_Q^2(t) dt \right\} p_{n-1}(x) \\ &\quad - \left\{ r_n \int_{-\infty}^{\infty} p_n(t) p_{n-1}(t) \bar{Q}(x, t) W_Q^2(t) dt \right\} p_n(x) \\ &= A_n(x) p_{n-1}(x) - B_n(x) p_n(x). \quad \blacksquare \end{aligned}$$

LEMMA 2.3. Let $|x| \leq \delta a_n$ ($0 < \delta < 1$). Then for a certain constant $c > 0$,

$$|p'_n(x)| W_Q(x) \leq c n a_n^{-3/2}$$

holds.

Proof. Since by [6, (12.21)] and [6, Theorem 12.3(b)] we have

$$A_n(x) \sim n/r_n \sim n/a_n \quad (|x| \leq 2a_n),$$

we see that by Schwarz's inequality

$$\begin{aligned} |B_n(x)| &\leq \left[r_n \int_{-\infty}^{\infty} p_n^2(t) \bar{Q}(x, t) W_Q^2(t) dt \right]^{1/2} \\ &\quad \times \left[r_n \int_{-\infty}^{\infty} p_{n-1}^2(t) \bar{Q}(x, t) W_Q^2(t) dt \right]^{1/2} \\ &\leq c n / a_n. \end{aligned}$$

On the other hand, by [6, Corollary 1.4],

$$|p_n(x)| W_Q(x), |p_{n-1}(x)| W_Q(x) \leq c a_n^{-1/2} \quad (|x| \leq \delta a_n).$$

Thus by Lemma 2.2 if $|x| \leq \delta a_n$, then

$$\begin{aligned} &|p'_n(x)| W_Q(x) \\ &\leq |A_n(x)| |p_{n-1}(x)| W_Q(x) + |B_n(x)| |p_n(x)| W_Q(x) \\ &\leq c n a_n^{-3/2}. \quad \blacksquare \end{aligned}$$

LEMMA 2.4. Let $|x_{jn}|, |x_{j-1, n}| \leq \delta a_n$ ($0 < \delta < 1$): and let

$$|p_n(\bar{x}_{jn})| = \max_{x_{jn} \leq x \leq x_{j-1, n}} |p_n(x)|, \quad x_{jn} < \bar{x}_{jn} < x_{j-1, n}. \quad (2.4)$$

Then we have

$$(i) \quad |p_n(\bar{x}_{jn})| W_Q(\bar{x}_{jn}) \sim a_n^{-1/2},$$

and

$$(ii) \quad |\bar{x}_{jn} - x_{jn}|, |\bar{x}_{jn} - x_{j-1, n}| \sim a_n/n, \quad (2.5)$$

that is,

$$(iii) \quad x_{j-1, n} - x_{jn} \sim a_n/n.$$

Proof. (i) By [6, Corollary 1.3], for $x_{jn} < x_{j,n+1} < x_{j-1,n}$, where $|x_{jn}|, |x_{j-1,n}| \leq \delta a_n$, we see

$$|p_n(x_{j,n+1})| W_Q(x_{j,n+1}) \sim a_n^{-1/2}. \quad (2.6)$$

On the other hand, [6, Corollary 1.4] means

$$|p_n(x)| W_Q(x) \leq ca_n^{-1/2} \quad (|x| \leq \delta a_n).$$

Therefore we have (i).

(ii) From (i) we see

$$\begin{aligned} ca_n^{-1/2}/(\bar{x}_{jn} - x_{jn}) &\leq |p_n(\bar{x}_{jn})| W_Q(\bar{x}_{jn})/(\bar{x}_{jn} - x_{jn}) \\ &= |p'_n(\xi)| W_Q(\bar{x}_{jn}) \quad (x_{jn} < \xi < \bar{x}_{jn}). \end{aligned} \quad (2.7)$$

Since by [6, Corollary 1.2 (b)] we see

$$|\bar{x}_{jn} - \xi| \leq \kappa a_n/n$$

for some $\kappa > 0$, Lemma 2.1 means $W_Q(\bar{x}_{jn}) \leq c W_Q(\xi)$. Thus from (2.7) and Lemma 2.3 we have

$$ca_n^{-1/2}/(\bar{x}_{jn} - x_{jn}) \leq c |p'_n(\xi)| W_Q(\xi) \leq cna_n^{-3/2},$$

that is,

$$ca_n/n \leq |\bar{x}_{jn} - x_{jn}|.$$

Consequently, we obtain $|\bar{x}_{jn} - x_{jn}| \sim a_n/n$, and similarly $|\bar{x}_{jn} - x_{j-1,n}| \sim a_n/n$.

(iii) It is trivial from (2.5). ■

The following lemma is important itself. It gives certain exact values of $p_n(x)$ in each interval

$$I_{jn}(\delta, \varepsilon) = [x_{jn} + \varepsilon a_n/n, x_{j-1,n} - \varepsilon a_n] \cap [-\delta a_n, \delta a_n]. \quad (2.8)$$

LEMMA 2.5. *Let $I_{jn}(\delta, \varepsilon)$ be defined in (2.8). For each $0 < \delta < 1$ there exists $\varepsilon > 0$ such that*

$$x_{jn} + \varepsilon a_n/n < x_{j-1,n} - \varepsilon a_n/n, \quad (2.9)$$

whenever $|x_{jn}|, |x_{j-1,n}| \leq \delta a_n$. Then

$$|p_n(x)| W_Q(x) \sim a_n^{-1/2}, \quad x \in I_{jn}(\delta, \varepsilon), \quad (2.10)$$

holds uniformly with respect to all $I_{jn}(\delta, \varepsilon) \neq \emptyset$.

Remark. We can also give certain exact values of $p'_n(x)$ in each interval

$$\bar{I}_{jn}(\delta, \varepsilon) = [x_{jn} - \varepsilon a_n/n, x_{jn} + \varepsilon a_n/n] \cap [-\delta a_n, \delta a_n].$$

For each $0 < \delta < 1$ there exists $\varepsilon > 0$ such that

$$|p'_n(x)| W_Q(x) \sim n a_n^{-3/2}, \quad x \in \bar{I}_{jn}(\delta, \varepsilon),$$

holds uniformly with respect to all $\bar{I}_{jn}(\delta, \varepsilon) \neq \phi$.

Proof of Lemma 2.5. (i) It follows from Lemma 2.4(iii) that there exists $\varepsilon > 0$ such that (2.9) holds for every j satisfying $|x_{jn}|, |x_{j-1,n}| \leq \delta a_n$. Let $I_{jn}(\delta, \varepsilon) \neq \phi$, then $\bar{x}_{jn} \in [x_{jn}, x_{j-1,n}]$ is defined in (2.4). In each interval (x_{jn}, \bar{x}_{jn}) or $(\bar{x}_{jn}, x_{j-1,n})$ the polynomial $p_n(x)$ has at most one inflection point. We shall first consider $p_n(x)$ in $[x_{jn}, \bar{x}_{jn}]$. Then we have one of two following cases.

(a) $|p_n(x)|$ is concave on $[x_{jn}, \bar{x}_{jn}]$.

(b) There exists $x_{jn} < x'_{jn} < \bar{x}_{jn}$ such that $|p_n(x)|$ is convex on $[x_{jn}, x'_{jn}]$, and concave on $[x'_{jn}, \bar{x}_{jn}]$.

Let us now define the line

$$y = \{ |p_n(\bar{x}_{jn})| / (\bar{x}_{jn} - x_{jn}) \} (x - x_{jn}).$$

Then for the case of (a) we see

$$\{ |p_n(\bar{x}_{jn})| / (\bar{x}_{jn} - x_{jn}) \} (x - x_{jn}) \leq |p_n(x)|, \quad x \in [x_{jn}, \bar{x}_{jn}]. \quad (2.11)$$

We shall treat the case of (b). If (2.11) is not correct, then we consider the line

$$y = |p'_n(x_{jn})| (x - x_{jn}),$$

and so we see

$$|p'_n(x_{jn})| (x - x_{jn}) \leq |p_n(x)|, \quad x \in [x_{jn}, \bar{x}_{jn}]. \quad (2.12)$$

Using Lemma 2.4(i), (ii) and Lemma 2.1, the inequality (2.11) means

$$c n a_n^{-3/2} (x - x_{jn}) \leq |p_n(x)| W_Q(x), \quad x \in [x_{jn}, \bar{x}_{jn}]. \quad (2.13)$$

If (2.12) holds, then by Lemma 2.1 and [6, Corollary 1.3] we also obtain (2.13). Hence if $x_{jn} + \varepsilon a_n/n \leq x \leq \bar{x}_{jn}$, then for a constant $c(\varepsilon)$

$$0 < c(\varepsilon) a_n^{-1/2} \leq |p_n(x)| W_Q(x). \quad (2.14)$$

On the other hand, from [6, Corollary 1.4]

$$|p_n(x)| W_Q(x) \leq ca_n^{-1/2}, \quad |x| \leq \delta a_n,$$

that is, with (2.14) we have for $x \in [x_{jn} + \varepsilon a_n/n, \bar{x}_{jn}]$

$$|p_n(x)| W_Q(x) \sim a_n^{-1/2}. \quad (2.15)$$

Similarly, for $x \in [\bar{x}_{jn}, x_{j-1,n} - \varepsilon a_n/n]$ we also have (2.15). Therefore, we obtain (2.10). ■

3. PROOF OF THEOREM

Now, we shall prove the theorem. The proof is along the same lines as Nevai's.

Proof of Theorem. We consider the space $C_0(-2, -1)$ which consists of continuous functions on \mathbb{R} with support in $[-2, -1]$. By our assumption, (1.2) is satisfied for $f \in C_0(-2, -1)$. Hence for the linear functional $L_n(f)$ on $C_0(-2, -1)$ we can apply the uniform boundedness theorem (cf. Theorem 10.19 of [11, p. 182]), and so for $f \in C_0(-2, -1)$ we have

$$\int_{-\infty}^{\infty} |L_n(f; x)|^p W(x) dx \leq c \max_{-2 \leq x \leq -1} |f(x)|^p. \quad (3.1)$$

Let $\{p_n(x)\}$ be the orthonormal polynomials with respect to the weight $W_Q^2(x)$, and for each $n=1, 2, 3, \dots$ let us consider a function $g_n \in C_0(-2, -1)$ such that

$$\max_{-2 \leq x \leq -1} |g_n(x)| = 1, \quad g_n(x_{kn}) = \text{sign } p'_n(x_{kn}), \quad x_{kn} \in (-2, -1).$$

Then we see

$$L_n(g_n; x) = p_n(x) \sum_{-2 \leq x_{kn} \leq -1} |p'_n(x_{kn})|^{-1} (x - x_{kn})^{-1}.$$

By [6, Corollary 1.3]

$$|p'_n(x_{kn})|^{-1} \sim n^{-1} a_n^{3/2}, \quad -2 \leq x_{kn} \leq -1,$$

and Lemma 2.4(iii)

$$\text{Num. } [\{k; x_{kn} \in [-2, -1]\}] \sim n/a_n,$$

where $\text{Num.}[S]$ denotes the number of elements of the set S . Thus for $x > 0$

$$\begin{aligned} |L_n(g_n; x)| &\geq c(1+x)^{-1} |p_n(x)| n^{-1} a_n^{3/2} n a_n^{-1} \\ &= c a_n^{1/2} (1+x)^{-1} |p_n(x)|. \end{aligned} \quad (3.2)$$

From (3.1) and (3.2),

$$\begin{aligned} A &= \limsup_{n \rightarrow \infty} \int_0^\infty |a_n^{1/2} p_n(x)/(1+x)|^p W(x) dx \\ &\leq c \limsup_{n \rightarrow \infty} \int_0^\infty |L_n(g_n; x)|^p W(x) dx \\ &\leq c \max_{-2 \leq x \leq -1} |g_n(x)|^p \\ &= c < \infty. \end{aligned} \quad (3.3)$$

Let us define

$$\begin{aligned} I_{jn} &= I_{jn}(\varepsilon) = [x_{jn} + \varepsilon a_n/n, x_{j-1, n} - \varepsilon a_n/n], \\ \bar{I}_{jn} &= \bar{I}_{jn}(\varepsilon) = [x_{j-1, n} - \varepsilon a_n/n, x_{j-1, n} + \varepsilon a_n/n], \quad j=2, 3, 4, \dots \end{aligned}$$

Let $0 < \delta < 1$, then, from (2.14),

$$|p_n(x)| W_Q(x) \geq c a_n^{-1/2}, \quad x \in I_{jn} \cap [0, \delta a_n],$$

where c is independent of n . Thus by (3.3)

$$c \sum_{j=2}^n \int_{I_{jn} \cap [0, \delta a_n]} |W_Q^{-1}(x)/(1+x)|^p W(x) dx \leq A. \quad (3.4)$$

Therefore, we also see that by exchanging n for $n+1$ in (3.4) we have

$$c \sum_{j=2}^{n+1} \int_{I_{j, n+1} \cap [0, \delta a_{n+1}]} |W_Q^{-1}(x)/(1+x)|^p W(x) dx \leq A. \quad (3.5)$$

Here, we can show that for a certain $\varepsilon > 0$,

$$\bar{I}_{jn}(\varepsilon) \cap [0, \delta a_n] \subset I_{j, n+1}(\varepsilon) \cap [0, \delta a_{n+1}]. \quad (3.6)$$

In fact, by (2.6) for $|x_{j,n+1}| \leq \delta a_n$ there exists $c > 0$ such that

$$\begin{aligned} ca_n^{-1/2} &\leq |p_n(x_{j,n+1}) W_Q(x_{j,n+1})| \\ &= |\{p_n(x_{j,n+1})/(x_{j,n+1} - x_{jn})\} W_Q(x_{j,n+1})| (x_{j,n+1} - x_{jn}) \\ &\leq c |p'_n(\xi) W_Q(\xi)| (x_{j,n+1} - x_{jn}) \\ &\leq cna_n^{-3/2}(x_{j,n+1} - x_{jn}) \end{aligned}$$

(see Lemma 2.3). Thus we see

$$ca_n/n \leq (x_{j,n+1} - x_{jn}),$$

consequently, we have

$$x_{j,n+1} - x_{jn}, x_{j-1,n} - x_{j,n+1} \sim a_n/n.$$

This means (3.6). Hence by (3.5) and (3.6)

$$\begin{aligned} \sum_{j=2}^n \int_{I_{jn} \cap [0, \delta a_n]} |W_Q^{-1}(x)/(1+x)|^p W(x) dx \\ \leq c \sum_{j=2}^{n+1} \int_{I_{j,n+1} \cap [0, \delta a_{n+1}]} |W_Q^{-1}(x)/(1+x)|^p W(x) dx \leq A. \end{aligned} \quad (3.7)$$

Using (3.4) and (3.7), we conclude

$$\int_0^{\delta a_n} |W_Q^{-1}(x)/(1+x)|^p W(x) dx \leq cA,$$

where c is independent of n . Therefore

$$\int_0^\infty |W_Q^{-1}(x)/(1+x)|^p W(x) dx \leq cA. \quad (3.8)$$

Similarly, we have

$$\int_{-\infty}^0 |W_Q^{-1}(x)/(1+|x|)|^p W(x) dx \leq cA. \quad (3.9)$$

Consequently, by (3.8) and (3.9) we have (1.3), that is, the theorem follows. ■

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